

Classification of low dimensional Lie super-bialgebras

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Abstract

A thorough analysis of Lie super-bialgebra structures on Lie super-algebras $osp(1;2)$ and super $e(2)$ is presented. Combined technique of computer algebraic computations and a subsequent identification of equivalent structures is applied. In all the cases Poisson-Lie brackets on supergroups are found. Possibility of quantizing them in order to obtain quantum groups is discussed. It turns out to be straightforward for all but one structures for super- $E(2)$ group.

1 Introduction

About 10 years of intensive research in quantum groups [1] [2] demonstrated richness of investigated mathematical structures. No wonder that except from efforts to find their physical applications many authors tried to classify them from a mathematical point of view. The classifications are based on many different ideas. We do not wish to provide exhaustive list of them but would like to mention some of well known examples. If quantum $SL(2)$ group is defined as one that preserves a non-degenerate bilinear form two inequivalent deformations appear [3]. More complicated analysis of the group $GL(3)$ leads to a conclusion that 26 inequivalent deformations exist in this case [4]. Ref. [5] contains a very recent attempt to classify quantum $SL(3)$ groups. A classification of all the 4×4 matrices satisfying Yang-Baxter equation was done in [6]. Later this result was used in order to list possible quantum deformations of the supergroup $GL(1,1)$ [7]. An interest in deformations of $D = 4$ relativistic symmetries led to a classification of deformations of Lorentz and Poincaré groups [8].

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From the early years of interest in quantum groups it is well known that they are closely related with notions of Lie bialgebra and Lie-Poisson group. Suppose that a parameter q is introduced such that the value $q = 1$ corresponds to undeformed universal enveloping algebra $U(\mathcal{G})$ or a classical commutative algebra of functions on a Lie group G (with the Lie algebra \mathcal{G}). Then the structure of Hopf algebra of the quantum group gives rise in the classical limit to extra structure on G or \mathcal{G} . G becomes a Poisson-Lie group and \mathcal{G} a Lie bialgebra [9]. These structures can be viewed as possible *directions* of quantum deformations. A fundamental fact about them is that all can be quantized i.e. they arise as the classical limit from a bona fide quantum deformation [10]. This suggests that an effective approach to classify quantum groups could be by classifying their classical limits and that seems to be much easier to be pulled off. The study of Lie bialgebras becomes in some cases even easier as they might be of coboundary type what happens if they can be described by means of a classical r -matrix satisfying (modified) classical Yang-Baxter equation. Cohomological properties of semisimple Lie groups imply that all related Lie bialgebras are in fact coboundaries [9]. More elaborate argument shows that the same is true in the case of inhomogeneous groups of space time symmetries for any signature of metric for dimensionality of space-time $D > 2$ [11]. Thus the problem of classification of Lie bialgebra structures contains as a subproblem a classification of solutions of (modified) classical Yang-Baxter equation which has been studied by many authors. It should be stressed however that in most cases Lie algebras admit bialgebra structures which are not coboundaries. Recently many low dimensional Lie algebras were investigated for bialgebra structures revealing a surprising number of possibilities. Two-dimensional Galilei algebra admits 9 inequivalent Lie bialgebra structures and only one out of them is a coboundary [13]. The central extension of Galilei algebra by the mass operator admits 26 inequivalent Lie bialgebra structures out of which 8 are coboundaries [14]. Similar analysis aimed mainly on the construction of quantum groups was performed before in the cases of Heisenberg-Weyl and oscillator Lie algebras [15]. The classification of the possible classical r -matrices (and automatically of Lie bialgebra structures) was done in the case of Lorentz and Poincaré algebras [12]. Classical r -matrices for $SL(3)$ were listed in [27]. Corresponding R -matrices satisfying Yang-Baxter equation were found in [28].

In this paper we will study quantum deformations of supergroups and their classical limits - Lie super-bialgebras. Quantum supergroups have been studied by many authors. Knowledge of R -matrices and low-dimensional representations of $osp(1|2)$, $su(1|1)$, $gl(2|1)$, $sl(1|2)$ has been used to construct integrable models [16]. Up to our knowledge no systematic investigation of Lie super-bialgebras has been yet undertaken. We decided to study two cases: $osp(1, 2)$ and supersymmetric extension of $e(2)$ algebra both treated as complex. The first one is interesting as it plays in the supersymmetric case a role analogous to $sl(2)$. Many papers devoted to quantum supergroups are based on a quantum deformation proposed by [18]. A natural question

is: are other deformations possible like it is in the case of $sl(2)$? The supersymmetric extension of $e(2)$ was chosen as $e(2)$ group is one which has a simple structure but admits several inequivalent Lie bialgebras [19] [20]. There are 4 of them with one being strictly speaking a 1-parameter family. Two structures are generated by classical r -matrices. All the four quantum deformations of the group $E(2)$ are known which could be helpful in finding quantum deformations in the supersymmetric case. Some quantum deformations of super- $e(2)$ algebra have been already discussed in the literature. They can be (in analogy to $e(2)$ case) obtained by means of a contraction procedure [21].

The problem of classifying Lie bialgebra structures for a given Lie algebra is mostly a computational one. We found it effective to use at various steps a computer. For a Lie algebra with N generators it is necessary to deal with equations containing the number of parameters which increases as N^3 . Some of the equations are linear but remaining (co-Jacobi identities) are quadratic. The complexity of the problem grows quickly with N . The biggest obstacle in continuing with this program for large N is primarily the fact that not everything can be done by computer. The most delicate part of the problem is to single out orbits under the action of the group of automorphisms of \mathcal{G} in a computer produced space of solutions.

A part of our results (a list of bialgebra structures but not of Poisson-Lie brackets) was presented in [22]. The first result is that in the $osp(1, 2)$ case all the Lie super-bialgebras are coboundaries. We found three independent classical r -matrices. One solution is quite obvious. As $sl(2)$ is a subalgebra of $osp(1, 2)$ any r -matrix for the former algebra satisfying (not modified!) classical YB equation is automatically a r -matrix for the latter. The second solution corresponds to the quantum deformation of $osp(1, 2)$ described in [18]. Relations for quantum $OSp(1, 2)$ are given in [29]. The third solution is a new one and requires a detailed study.

In the case of super- $e(2)$ we found six families of independent Lie super-bialgebra structures. In their most general form they are not coboundaries but four of them contain coboundary members. Some links to the classification of analogous structures on $e(2)$ can be established. The case (i) in our list is in a clear analogy with δ_2 of [20]. The case (iv) corresponds to δ_1 . It is the infinitesimal form of quantum deformation introduced in [21]. The existence of r -matrix r_2 is again obvious as it consists only of generators of $e(2)$ and satisfies (not modified!) classical YB equation.

For all the Lie super-bialgebra structures we calculated corresponding Lie-Poisson brackets. Their form is such that in five cases of super $E(2)$ group it is easy to go through with the program of constructing quantum supergroups [24] [25]. It is sufficient to change Lie-Poisson brackets into (anti) commutators.

The paper is organized as follows. In Chapter 2 all the basic concepts and notation used in the rest of the paper are introduced. Chapters 3 and 4 contain derivation of inequivalent Lie super-bialgebras for $osp(1, 2)$ and

super- $e(2)$. In Chapter 5 Poisson-Lie brackets for both cases are presented. In Chapter 6 our conclusions and some final remarks are presented.

After completing our study we have learned about the paper in which by applying Drinfeld twisting procedure new quantum supergroup structure on $OSp(1, 2)$ was found [23]. It is clear that it corresponds to the above mentioned "trivial" Lie super-bialgebra structure (number 1 on our list presented in Chapter 4).

2 Basic definitions and notation

Super Lie-algebra \mathcal{G} is a graded vector space

$$\mathcal{G} = \mathcal{G}_B \oplus \mathcal{G}_F \quad (1)$$

with the grade function $grade(\mathcal{G}_B) = 0$, $grade(\mathcal{G}_F) = 1$. A Lie superalgebra structure is provided by a linear mapping

$$[,] : \mathcal{G} \otimes \mathcal{G} \rightarrow \mathcal{G} \quad (2)$$

satisfying requirements of (graded) antisymmetry and Jacobi identity. In order to express them it is useful to introduce a basis in \mathcal{G} $\{g_i\} \subset \mathcal{G}_B \cup \mathcal{G}_F$ and structure constants

$$[g_i, g_j] = c_{ij}^k g_k. \quad (3)$$

Structure constants have to satisfy

$$c_{ij}^k = 0 \quad \text{whenever} \quad grade(g_i) + grade(g_j) \neq grade(g_k) \pmod{2} \quad (4)$$

$$c_{ij}^k = -z(i, j)c_{ji}^k, \quad (5)$$

and

$$c_{ij}^k c_{kl}^m z(i, l) + c_{jl}^k c_{ki}^m z(j, i) + c_{li}^k c_{kj}^m z(l, j) = 0 \quad (6)$$

where

$$z(i, j) = (-1)^{grad(g_i)grad(g_j)}. \quad (7)$$

Lie super-bialgebra structure is a linear mapping

$$\delta : \mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{G} \quad (8)$$

which in the chosen basis reads

$$\delta(g_i) = f_i^{kl} g_k \otimes g_l. \quad (9)$$

δ has to satisfy several requirements. First of all it makes the dual linear space \mathcal{G}^* a Lie super-algebra

$$f_k^{ij} = 0 \quad \text{whenever} \quad grad(g_i) + grad(g_j) \neq grad(g_k) \pmod{2} \quad (10)$$

$$f_k^{ij} = -z(i, j) f_k^{ji} \quad (11)$$

$$f_i^{kj} f_j^{lm} z(k, m) + f_i^{lj} f_j^{mk} z(l, k) + f_i^{mj} f_j^{kl} z(m, l) = 0. \quad (12)$$

Moreover structure constants c 's and f 's have to be related

$$c_{ij}^k f_k^{lm} = f_i^{lk} c_{kj}^m + c_{kj}^l f_i^{km} z(m, j) + c_{jk}^l f_j^{km} + f_j^{lk} c_{ik}^m z(i, l) \quad (13)$$

Coboundary Lie super-bialgebra is a pair (\mathcal{G}, r) , where \mathcal{G} is a Lie super-bialgebra and $r \in \mathcal{G}_B \wedge \mathcal{G}_B \oplus \mathcal{G}_F \wedge \mathcal{G}_F \subset \mathcal{G} \wedge \mathcal{G}$. such that for every $g_i \in \mathcal{G}$

$$\delta(g_i) = [r, g_i \otimes 1 + 1 \otimes g_i] \quad (14)$$

Schouten bracket is defined as follows

$$[[r, r]] \equiv [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \quad (15)$$

where $r_{12} = r \otimes 1$, $r_{23} = 1 \otimes r$, ...

r satisfies classical Yang-Baxter equation (CYBE) if

$$[[r, r]] = 0 \quad (16)$$

and modified CYBE if

$$\forall g_i \in \mathcal{G} \quad [[r, r], g_i \otimes 1 \otimes 1 + 1 \otimes g_i \otimes 1 + 1 \otimes 1 \otimes g_i] = 0. \quad (17)$$

Super $e(2)$ Lie algebra is spanned by the set of generators $\{H, P_+, P_-, D_+, D_-\}$ which fulfill the following (anti) commutation relations

$$\begin{aligned} [H, P_{\pm}] &= \pm P_{\pm}, \\ [H, D_{\pm}] &= \pm \frac{1}{2} D_{\pm}, \\ [P_+, P_-] &= 0, \\ \{D_+, D_-\} &= 0, \\ \{D_{\pm}, D_{\pm}\} &= P_{\pm}, \\ [P_{\pm}, D_{\pm}] &= 0, \\ [P_{\pm}, D_{\mp}] &= 0, \end{aligned} \quad (18)$$

A convenient parameterization of the classical super- $E(2)$ group is obtained by means of exponentiation

$$g(s, a, b, \xi, \eta) = \exp(sH) \exp(aP_+) \exp(bP_-) \exp(\xi D_+) \exp(\eta D_-). \quad (19)$$

Coproducts are

$$\Delta(s) = s \otimes 1 + 1 \otimes s, \quad (20)$$

$$\Delta(a) = 1 \otimes a + a \otimes \exp(-s) + \frac{1}{2} \xi \otimes \xi \exp(-\frac{s}{2}), \quad (21)$$

$$\Delta(b) = 1 \otimes b + b \otimes \exp(s) + \frac{1}{2}\eta \otimes \eta \exp(\frac{s}{2}), \quad (22)$$

$$\Delta(\xi) = 1 \otimes \xi + \xi \otimes \exp(-\frac{s}{2}), \quad (23)$$

$$\Delta(\eta) = 1 \otimes \eta + \eta \otimes \exp(\frac{s}{2}). \quad (24)$$

Lie superalgebra $osp(1, 2)$ is spanned by the set of generators $\{H, X_+, X_-, V_+, V_-\}$ which fulfill the following (anti) commutation relations

$$\begin{aligned} [H, X_{\pm}] &= \pm X_{\pm}, \\ [H, V_{\pm}] &= \pm \frac{1}{2}V_{\pm}, \\ [X_+, X_-] &= 2H, \\ \{V_+, V_-\} &= -\frac{1}{2}H, \\ \{V_{\pm}, V_{\pm}\} &= \pm \frac{1}{2}X_{\pm}, \\ [X_{\pm}, V_{\pm}] &= 0, \\ [X_{\pm}, V_{\mp}] &= V_{\pm}, \end{aligned} \quad (25)$$

In the case of supergroup $OSp(1|2)$ we will use more implicit parameterization by means of 3×3 supermatrices subject to certain constraints [18]

$$g = \begin{pmatrix} a & \alpha & b \\ \gamma & e & \beta \\ c & \delta & d \end{pmatrix} \quad (26)$$

where $e = 1 + \alpha\delta$, $\gamma = c\alpha - a\delta$, $\beta = d\alpha - b\delta$, $ad - bc + \alpha\delta = 1$. Variables denoted by Greek letters are of Grassmanian type. Coproducts follow from matrix multiplication of elements of G .

3 Lie super-bialgebras for $osp(1, 2)$

The problem studied in Chapters 3 and 4 can be formulated in the following way:

Given a set of structure constants c_{ij}^k find all sets of structure constants f_m^{np} that give rise to Lie super-bialgebras. Two such sets are considered equivalent if they can be made equal by a change of the basis in \mathcal{G} .

Verify if obtained structures are or not of coboundary type.

Initial steps in the analysis can be made using a computer. An arbitrary form of f 's satisfying super-antisymmetry and preserving the grading is assumed. Constraints from the set of linear equations coming from the co-cycle condition (13) are firstly taken into account. Then the set of quadratic equations coming from the super co-Jacobi conditions (12) is to be solved.

In the case of $osp(1, 2)$ we discover that all the possible bialgebra structures are coboundaries and that the classical r -matrix is in one of two possible

forms:

$$\begin{aligned}
r_a = & \ x(X_+ \wedge X_- + 2V_+ \wedge V_-) \\
& + y(H \wedge X_+ - V_+ \wedge V_+) \\
& + z(H \wedge X_- - V_- \wedge V_-),
\end{aligned} \tag{27}$$

$$r_b = \pm \sqrt{uv} X_+ \wedge X_- + uH \wedge X_+ + vH \wedge X_- ,$$

where x, y, z, u, v are arbitrary complex numbers.

Identification of automorphisms of superalgebra $osp(1, 2)$ is fairly straightforward. They can not mix fermions with bosons (grading is preserved). As the superalgebra is generated just by the two fermions V_+ and V_- every automorphism is determined by the following transformation of the fermions;

$$\begin{aligned}
\tilde{V}_+ &= a V_+ + b V_- \\
\tilde{V}_- &= c V_+ + d V_- .
\end{aligned} \tag{28}$$

We easily derive that under the above transformation

$$\begin{aligned}
\tilde{H} &= -ac X_+ + (ad + bc) H + bd X_- , \\
\tilde{X}_+ &= a^2 X_+ - 2ab H - b^2 X_- , \\
\tilde{X}_- &= -c^2 X_+ + 2cd H + d^2 X_- .
\end{aligned} \tag{29}$$

The operators $\{\tilde{H}, \tilde{X}_+, \tilde{X}_-, \tilde{V}_+, \tilde{V}_-\}$ obey the same super-commutation relations as $\{H, X_+, X_-, V_+, V_-\}$ if and only if

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 . \tag{30}$$

Under the transformation (28)-(29) the parameters x, y, z of the r -matrix r_a given in (27) transform as follows

$$\begin{pmatrix} \tilde{y} & -\tilde{x} \\ -\tilde{x} & \tilde{z} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y & -x \\ -x & z \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} . \tag{31}$$

We notice that this is exactly the way the symmetric form transforms under the change of basis. If we take into account the Sylvester theorem we see that we can make the matrix

$$\begin{pmatrix} y & -x \\ -x & z \end{pmatrix}$$

diagonal ($x = 0$) with $y = z$ or $y = 1, z = 0$ (without the condition (30) it would be possible to make either $y = z = 1$ or $y = 1, z = 0$).

Thus r_a if different from zero is equivalent to one of the following:

$$r_a = \begin{cases} r_2 \equiv H \wedge X_+ - V_+ \wedge V_+ & \text{if } x^2 - yz = 0 \\ r_3 \equiv t(H \wedge X_+ - V_+ \wedge V_+ + H \wedge X_- - V_- \wedge V_-) & \text{if } x^2 - yz \neq 0 . \end{cases} \tag{32}$$

By means of analogous reasoning we arrive at the conclusion that the r -matrix r_b is always equivalent to

$$r_1 \equiv H \wedge X_+ . \tag{33}$$

4 Lie super-bialgebras for super- $e(2)$

The initial steps of the analysis are made using a computer like in the case of $osp(1, 2)$. After solving equations quadratic in structure constants it turns out that most of bialgebra structures are not coboundaries. This makes the investigation more involved since we cannot use the r -matrix formulation and must explicitly state the co-Lie structure for each solution found.

The possibilities found by means of the computer are:

Case **A**

$$\begin{aligned}
\delta(H) &= H \wedge (aP_+ + bP_-) + cP_+ \wedge P_- , \\
\delta(P_+) &= P_+ \wedge bP_- , \\
\delta(P_-) &= -aP_+ \wedge P_- , \\
\delta(D_+) &= \frac{1}{2}(aP_+ - bP_-) \wedge D_+ \pm \sqrt{ab}P_+ \wedge D_- , \\
\delta(D_-) &= \frac{1}{2}(aP_+ - bP_-) \wedge D_- \pm \sqrt{ab}P_- \wedge D_+ ,
\end{aligned} \tag{34}$$

Case **B**

$$\begin{aligned}
\delta(H) &= a(H \wedge P_+ - \frac{1}{2}D_+ \wedge D_+) \\
&\quad + b(H \wedge P_- + \frac{1}{2}D_- \wedge D_-) + cP_+ \wedge P_- , \\
\delta(P_+) &= P_+ \wedge bP_- + d(2H \wedge P_+ - D_+ \wedge D_+) , \\
\delta(P_-) &= -aP_+ \wedge P_- + d(2H \wedge P_- + D_- \wedge D_-) , \\
\delta(D_+) &= -\frac{1}{2}(aP_+ + bP_-) \wedge D_+ + d(H \wedge D_+) , \\
\delta(D_-) &= -\frac{1}{2}(aP_+ + bP_-) \wedge D_- + d(H \wedge D_-) ,
\end{aligned} \tag{35}$$

where a, b, c, d are arbitrary complex numbers such that $cd = 0$.

The set of automorphisms of super- $e(2)$ is generated by three transformations:

1. $\widetilde{H} = H + \alpha P_+ + \beta P_- , \quad \widetilde{P}_\pm = P_\pm , \quad \widetilde{D}_\pm = D_\pm ,$
2. $\widetilde{H} = -H , \quad \widetilde{P}_\pm = P_\mp , \quad \widetilde{D}_\pm = D_\mp ,$
3. $\widetilde{H} = H ,$
 $\widetilde{P}_+ = \alpha^2 P_+ , \quad \widetilde{D}_+ = \alpha D_+ ,$
 $\widetilde{P}_- = \beta^2 P_- , \quad \widetilde{D}_- = \beta D_- ,$

When taken into account they lead to a conclusion that a and b can be scaled out to take value of 1 or 0. It is also possible to show that when $d \neq 0$ then by taking $\widetilde{H} = H + \frac{a}{2d}P_+ + \frac{b}{2d}P_-$, we can make $a = b = c = 0$.

Finally we arrive at six families of bialgebra structures whose representatives can be described by means of the following substitutions:

- (i) Case **A** where $a = b = 0$,
- (ii) Case **A** where $a = 1, b = 0$,
- (iii) Case **A** where $a = b = 1$,

- (iv) Case **B** where $a = b = c = 0$,
- (v) Case **B** where $a = 1, b = d = 0$,
- (vi) Case **B** where $a = b = 1, d = 0$.

Classical r -matrices exist if both c and d vanish. Their general forms for cases **A** and **B** are

$$\begin{aligned} r_A &= aH \wedge P_+ - bH \wedge P_- + \sqrt{ab}D_+ \wedge D_- + fP_+ \wedge P_- , \\ r_B &= a(H \wedge P_+ - \frac{1}{2}D_+ \wedge D_+) - b(H \wedge P_- + \frac{1}{2}D_- \wedge D_-) + fP_+ \wedge P_- , \end{aligned} \quad (36)$$

Term $P_+ \wedge P_-$ is irrelevant and will be omitted since its commutators (14) with all the generators of super- $e(2)$ vanish.

After substitutions we obtain

$$\begin{aligned} r_{(ii)} &= H \wedge P_+ , \\ r_{(iii)} &= H \wedge P_+ - H \wedge P_- + D_+ \wedge D_- , \\ r_{(v)} &= H \wedge P_+ - \frac{1}{2}D_+ \wedge D_+ , \\ r_{(vi)} &= H \wedge P_+ - \frac{1}{2}D_+ \wedge D_+ - H \wedge P_- - \frac{1}{2}D_- \wedge D_- , \end{aligned} \quad (37)$$

Only $r_{(ii)}$ and $r_{(v)}$ satisfy CYBE.

5 Poisson-Lie brackets

Lie super-bialgebras are in 1 : 1 correspondence with Poisson-Lie structures on supergroups [26]. Poisson-Lie brackets satisfy the following properties:

$$\{f, g\} = -z(f, g)\{g, f\} , \quad (38)$$

$$\{f, gh\} = \{f, g\}h + z(f, g)g\{f, h\} , \quad (39)$$

$$\begin{aligned} 0 &= z(f, h)\{f, \{g, h\}\} + z(g, f)\{g, \{h, f\}\} \\ &\quad + z(h, g)\{h, \{f, g\}\} , \end{aligned} \quad (40)$$

$$\Delta\{f, g\} = \{\Delta f, \Delta g\} . \quad (41)$$

Poisson-Lie brackets are most easily introduced by means of left- and right-invariant vector fields on a supergroup. In the super case one should distinguish left- and right-hand side derivatives of superfunctions. In coboundary case with a classical r -matrix r_{kl} Poisson-Lie brackets are given by [25]

$$\{\phi, \psi\} = (Y_k^{(r)}\phi) r_{kj} (Y_j^{(l)}\psi) - (X_k^{(r)}\phi) r_{kj} (X_j^{(l)}\psi) \quad (42)$$

where $Y_k^{(r,l)}$ denotes left-invariant right (r) or left (l) derivatives and $X_k^{(r,l)}$ right-invariant right (r) or left (l) derivatives. Y_k and X_k can be derived from the coproducts.

We present below how they act on generators of both supergroups.

a) super- $E(2)$

$$Y_H^{(r,l)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -a \\ b \\ 1 \\ -\xi/2 \\ \eta/2 \end{pmatrix}, \quad X_H^{(r,l)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad (43)$$

$$Y_{P_+}^{(r,l)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad X_{P_+}^{(r,l)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \exp(-s) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (44)$$

$$Y_{P_-}^{(r,l)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad X_{P_-}^{(r,l)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ \exp(s) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (45)$$

$$Y_{D_-}^{(r)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ \eta/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad X_{D_-}^{(r)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ -\eta \exp(\frac{s}{2})/2 \\ 0 \\ 0 \\ \exp(\frac{s}{2}) \end{pmatrix}, \quad (46)$$

$$Y_{D_-}^{(l)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ -\eta/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad X_{D_-}^{(l)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ \eta \exp(\frac{s}{2})/2 \\ 0 \\ 0 \\ \exp(\frac{s}{2}) \end{pmatrix}, \quad (47)$$

$$Y_{D_+}^{(r)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi/2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad X_{D_+}^{(r)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -\xi \exp(-\frac{s}{2})/2 \\ 0 \\ 0 \\ \exp(-\frac{s}{2}) \\ 0 \end{pmatrix}, \quad (48)$$

$$Y_{D_+}^{(l)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -\xi/2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad X_{D_+}^{(l)} \begin{pmatrix} a \\ b \\ s \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \exp(-\frac{s}{2})/2 \\ 0 \\ 0 \\ \exp(-\frac{s}{2}) \\ 0 \end{pmatrix}. \quad (49)$$

b) $OSp(1|2)$

$$Y_H^{(r,l)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} a/2 \\ 0 \\ -b/2 \\ c/2 \\ 0 \\ -d/2 \end{pmatrix}, \quad X_H^{(r,l)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} a/2 \\ \alpha/2 \\ b/2 \\ -c/2 \\ -\delta/2 \\ -d/2 \end{pmatrix}, \quad (50)$$

$$Y_{X_+}^{(r,l)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \\ c \end{pmatrix}, \quad X_{X_+}^{(r,l)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} c \\ \delta \\ d \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (51)$$

$$Y_{X_-}^{(r,l)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ 0 \\ d \\ 0 \\ 0 \end{pmatrix}, \quad X_{X_-}^{(r,l)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a \\ \alpha \\ b \end{pmatrix}, \quad (52)$$

$$Y_{V_+}^{(r)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ a/2 \\ \alpha/2 \\ 0 \\ c/2 \\ \delta/2 \end{pmatrix}, \quad X_{V_+}^{(r)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} -\gamma/2 \\ e/2 \\ -\beta/2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (53)$$

$$Y_{V_+}^{(l)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ a/2 \\ -\alpha/2 \\ 0 \\ c/2 \\ -\delta/2 \end{pmatrix}, \quad X_{V_+}^{(l)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} \gamma/2 \\ e/2 \\ \beta/2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (54)$$

$$Y_{V_-}^{(r)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} -\alpha/2 \\ b/2 \\ 0 \\ -\delta/2 \\ d/2 \\ 0 \end{pmatrix}, \quad X_{V_-}^{(r)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\gamma/2 \\ e/2 \\ -\beta/2 \end{pmatrix}, \quad (55)$$

$$Y_{V_-}^{(l)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} \alpha/2 \\ b/2 \\ 0 \\ \delta/2 \\ d/2 \\ 0 \end{pmatrix}, \quad X_{V_-}^{(l)} \begin{pmatrix} a \\ \alpha \\ b \\ c \\ \delta \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \gamma/2 \\ e/2 \\ \beta/2 \end{pmatrix}. \quad (56)$$

In non-coboundary cases Poisson brackets can be calculated by applying supersymmetric version of the method described in [20]. It is necessary to solve the cocycle equation for an element $\Phi : G \rightarrow \mathcal{G} \wedge \mathcal{G}$

$$\Phi(g_1 g_2) = \Phi(g_1) + g_1 \Phi(g_2) g_1^{-1} \quad (57)$$

satisfying "initial conditions" determined by the Lie super-bialgebra structure in consideration.

The whole procedure has to be followed in every detail only in the case of Lie super-bialgebra structures (i) and (iv) from the list presented in Chapter 3. One finds

$$\Phi_{(i)}(g) = csP_+ \wedge P_- \quad (58)$$

$$\begin{aligned} \Phi_{(iv)}(g) = & -2ae^s P_+ \wedge H - ae^s D_+ \wedge D_+ - 2be^{-s} P_- \wedge H \\ & + 2abP_- \wedge P_+ + be^{-s} D_- \wedge D_- + \xi e^{\frac{s}{2}} H \wedge D_+ \\ & - a\xi e^{\frac{3s}{2}} P_+ \wedge D_+ + \xi be^{-\frac{s}{2}} P_- \wedge D_+ + \eta e^{-\frac{s}{2}} H \wedge D_- \\ & - a\eta e^{\frac{s}{2}} P_+ \wedge D_- + \eta be^{-\frac{3s}{2}} P_- \wedge D_- - \frac{1}{2}\xi\eta D_+ \wedge D_- . \end{aligned} \quad (59)$$

For $\Phi(g) = \Phi^{jk}(g)g_j \wedge g_k$ Poisson-Lie brackets are

$$\{\phi, \psi\} = (X_j^{(r)} \phi) \Phi^{jk} (X_k^{(l)} \psi). \quad (60)$$

In all the remaining cases one can use a fact that for the special value of parameter c : $c = 0$ they become coboundaries. The parameter c is present only in $\delta(H)$ as $cP_+ \wedge P_-$. It is clear that it will appear in Φ as $csP_+ \wedge P_-$. It is thus possible to calculate Poisson-Lie brackets using classical r -matrices given in (37) and to add at the very end of computations the extra term in $\{a, b\}$. The complete set of relations making $OSp(1, 2)$ and super $E(2)$ Poisson-Lie supergroups are given below in two tables. In the case of $OSp(1, 2)$ in order to obtain shorter formulae all the Poisson brackets have been multiplied by the factor 2.

Table 1: Poisson Lie structures for the group $OSp(1|2)$.

	1	2	3
$\{a, b\}$	$a^2 + \alpha\delta - 1$	$a^2 - 1$	$a^2 + b^2 - 1$
$\{a, c\}$	$-c^2$	$-c^2$	$1 - c^2 - a^2$
$\{a, d\}$	$c(a - d)$	$c(a - d)$	$(c - b)(a - d)$
$\{b, c\}$	$-c(a + d)$	$-c(a + d)$	$-(b + c)(a + d)$
$\{b, d\}$	$1 - d^2 - \alpha\delta$	$1 - d^2$	$1 - b^2 - d^2$
$\{c, d\}$	c^2	c^2	$c^2 + d^2 - 1$
$\{a, \alpha\}$	$c\alpha - a\delta$		$b\alpha$
$\{b, \alpha\}$	$d\alpha - b\delta$	$-a\alpha$	$-a\alpha$
$\{c, \alpha\}$	$c\delta$	$c\delta$	$c\delta + a\alpha + b\delta$
$\{d, \alpha\}$	$d\delta$	$(d - a)\delta$	$d\delta + b\alpha - a\delta$
$\{a, \delta\}$	$-c\delta$	$-c\delta$	$-c\delta - a\alpha + d\alpha$
$\{b, \delta\}$	$-d\delta$	$-d\delta - c\alpha$	$-(d\delta + b\alpha + c\alpha)$
$\{c, \delta\}$			$d\delta$
$\{d, \delta\}$		$-c\delta$	$-c\delta$
$\{\alpha, \alpha\}$	$2\delta\alpha$	$1 - a^2$	$1 - a^2 - b^2$
$\{\alpha, \delta\}$		$-ac$	$-ac - bd$
$\{\delta, \delta\}$		$-c^2$	$1 - c^2 - d^2$

 Table 2: Poisson Lie structures for the group super- $E(2)$.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
$\{a, b\}$	cs	$-b + cs$	$a - b + cs$	$-2ab$	$-b - e^s$	$a - b + cs$
$\{a, e^s\}$		$1 - e^s$	$1 - e^s$	$-2ae^s$	$1 - e^s$	$1 - e^s$
$\{b, e^s\}$			$e^s - e^{2s}$	$-2be^s$		$e^s - e^{2s}$
$\{a, \xi\}$		$\xi/2$	$\xi/2$		$-\xi e^{-s}/2$	$-\xi e^{-s}/2$
$\{a, \eta\}$		$-\eta/2$	$\xi - (\eta/2)$	$-a\eta$	$-\eta/2$	$-\eta/2$
$\{b, \xi\}$			$\eta - (\xi/2)$	ξb		$-\xi/2$
$\{b, \eta\}$			$\eta/2$			$-\eta e^s/2$
$\{e^s, \xi\}$				ξe^s		
$\{e^s, \eta\}$				ηe^s		
$\{\xi, \xi\}$				$-2a$	$e^{-s} - 1$	$e^{-s} - 1$
$\{\xi, \eta\}$				$-\xi\eta/2$		
$\{\eta, \eta\}$				$2b$		$e^s - 1$

6 Discussion and final remarks

In five cases of Poisson-Lie brackets on the super- $E(2)$ group it is straightforward to obtain quantum group structures. In fact, RHS's of Poisson brackets are rather simple and contain no products of functions which could lead to ordering ambiguities upon quantization. Deformation of super- $E(2)$ introduced in [21] corresponds in our classification to the structure (iv). By applying duality techniques in cases (i), (ii), (iii), (v) and (vi) it must be possible to

obtain quantum deformations of super $e(2)$ algebra. In the case of $OSp(1, 2)$ a quantization of the first structure appeared very recently in [23]. The second structure is the standard one introduced in [18]. A quantum version of the third structure is missing.

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